Euler's Pentagonal Number Theorem, An Introduction to Integer Partitions

Eli DeWitt

AIMS Summer School

July 17, 2026

Opening Comments

- Thanks to Jon for inviting me to give this talk.
- Thank YOU for coming to my talk to learn more about integer partitions.

Integer Partitions

A **partition of n** is a finite collection of positive integers whose sum is n.

Integer Partitions

A **partition of n** is a finite collection of positive integers whose sum is n.

Example: Here are all of the partitions of 4:

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$

The summands are called **parts**.

Integer Partitions

A **partition of n** is a finite collection of positive integers whose sum is n.

Example: Here are all of the partitions of 4:

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$

The summands are called **parts**.

We define the partition function $p(n)$ to count the number of partitions of n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Integer Partitions

A **partition of n** is a finite collection of positive integers whose sum is n.

Example: Here are all of the partitions of 4:

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$

The summands are called **parts**.

We define the partition function $p(n)$ to count the number of partitions of n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Question. Is there a nice formula for $p(n)$?

Integer Partitions

Theorem (Rademacher, 1937)

The following series converges to $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}}\sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right]\right)$$

where

$$A_k(n) = \sum_{0\le m<k,(m,k)=1} \exp\left(\pi i\left(s(m,k) - 2n\frac{m}{k}\right)\right)$$

with

$$s(m,k) = \frac{1}{4k}\sum_{j=1}^{k-1} \cot\left(\frac{\pi j}{k}\right)\cot\left(\frac{\pi jm}{k}\right)$$

Integer Partitions

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Integer Partitions

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Question. Can we find a (nice) formula for $p(n)$?

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Question. Can we find a (nice) formula for $p(n)$?

Our goal is to visit some of the combinatorial tools used to study partitions, then land on a surprising recurrence formula involving pentagonal numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table: Partition function up to $n = 14$

Question. Can we find a (nice) formula for $p(n)$?

Our goal is to visit some of the combinatorial tools used to study partitions, then land on a surprising recurrence formula involving pentagonal numbers.

A surprisingly useful combinatorial tool is the **partition generating function**:

$$\sum_{n=0}^{\infty} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \dots$$

Generating Functions

Consider the infinite product

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+x^{4n}+\dots\right)$$

Consider the infinite product

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+x^{4n}+\ldots\right)$$

$$\begin{aligned}
&=(1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots)\\
&\times(1+x^{2}+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots)\\
&\times(1+x^{3}+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\
&\times(1+x^{4}+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+\cdots)
\end{aligned}$$

Consider the infinite product

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+x^{4n}+\dots\right)$$

$$\begin{aligned}
&=\left(1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots\right)\\
&\times\left(1+x^{2}+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots\right)\\
&\times\left(1+x^{3}+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots\right)\\
&\times\left(1+x^{4}+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+\cdots\right)
\end{aligned}$$

$$=1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^{2}+x^{2+1}+x^{2+1+1}+x^{2+2}+x^{3}+x^{3+1}+\cdots$$

Generating Functions

Consider the infinite product

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+x^{4n}+\dots\right)$$

$$\begin{aligned}
&= (1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots)\\
&\times(1+x^{2}+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots)\\
&\times(1+x^{3}+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\
&\times(1+x^{4}+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+\cdots)
\end{aligned}$$

$$= 1+x^{1}+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^{2}+x^{2+1}+x^{2+1+1}+x^{2+2}+x^{3}+x^{3+1}+\cdots$$

Each factor in the "chooses" how many parts of size n appear in a partition.

Consider the infinite product

$$\prod_{n=1}^{\infty}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\dots\right)$$

$$\begin{aligned}&=(1+x^1+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots)\\&\times(1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots)\\&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\&\times(1+x^4+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+\cdots)\end{aligned}$$

$$=1+x^1+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^2+x^{2+1}+x^{2+1+1}+x^{2+2}+x^3+x^{3+1}+\cdots$$

Each factor in the "chooses" how many parts of size n appear in a partition.
Each partition contributes $+1$ to a count of partitions of n.

Consider the infinite product

$$\prod_{n=1}^{\infty} \left(1 + x^n + x^{2n} + x^{3n} + x^{4n} + \dots\right)$$

$$\begin{aligned}
&= (1 + x^1 + x^{1+1} + x^{1+1+1} + x^{1+1+1+1} + \cdots) \\
&\times(1 + x^2 + x^{2+2} + x^{2+2+2} + x^{2+2+2+2} + \cdots) \\
&\times(1 + x^3 + x^{3+3} + x^{3+3+3} + x^{3+3+3+3} + \cdots) \\
&\times(1 + x^4 + x^{4+4} + x^{4+4+4} + x^{4+4+4+4} + \cdots)
\end{aligned}$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Each factor in the "chooses" how many parts of size n appear in a partition.
Each partition contributes $+1$ to a count of partitions of n.

Each factor is a geometric series

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+x^{4n}+\dots\right)=\prod_{n=1}^{\infty}\frac{1}{1-x^{n}}$$

Each factor is a geometric series

$$\prod_{n=1}^{\infty}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\dots\right)=\prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

Theorem

For all $|x|<1$,

$$\sum_{n=1}^{\infty}p(n)x^n=\prod_{n=1}^{\infty}\frac{1}{1-x^n}.$$

Generating Functions

Each factor is a geometric series

$$\prod_{n=1}^{\infty}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\dots\right)=\prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

Theorem

For all $|x|<1$,

$$\sum_{n=1}^{\infty}p(n)x^n=\prod_{n=1}^{\infty}\frac{1}{1-x^n}.$$

Question. What is that denominator?

$$\left(\sum_{n=1}^{\infty}p(n)x^n\right)^{-1}=\prod_{n=1}^{\infty}(1-x^n)$$

Each factor is a geometric series

$$\prod_{n=1}^{\infty}\left(1+x^n+x^{2n}+x^{3n}+x^{4n}+\dots\right)=\prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

Theorem

For all $|x|<1$,

$$\sum_{n=1}^{\infty}p(n)x^n=\prod_{n=1}^{\infty}\frac{1}{1-x^n}.$$

Question. What is that denominator? Partitions of n into distinct parts!

$$\left(\sum_{n=1}^{\infty}p(n)x^n\right)^{-1}=\prod_{n=1}^{\infty}(1-x^n)$$

$$\left(\sum_{n=1}^{\infty} p(n)x^n\right)^{-1} = \prod_{n=1}^{\infty}(1-x^n)$$

After expanding the product, each partition of n into k distinct parts will correspond to a term $(-1)^k x^n$.

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=0}^{\infty}(d_e(n) - d_o(n))x^n$$

where $d_e(n)$ (resp. $d_o(n)$) count the number of partitions of n into an even (resp. odd) number of distinct parts.

Review. By translating into the world of infinite products, we discovered that the reciprocal of the partition generating function is a new generating function which detects a balance of distinct partitions using the parity of the number of parts.

$$\left(\sum_{n=1}^{\infty} p(n)x^n\right)^{-1} = \sum_{n=1}^{\infty}(d_e(n) - d_o(n))x^n$$

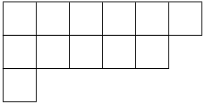
Distinct Parts and Ferrers Diagrams

Review. By translating into the world of infinite products, we discovered that the reciprocal of the partition generating function is a new generating function which detects a balance of distinct partitions using the parity of the number of parts.

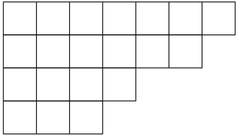
$$\left(\sum_{n=1}^{\infty} p(n)x^n\right)^{-1} = \sum_{n=1}^{\infty}(d_e(n) - d_o(n))x^n$$

To study partitions into distinct parts, we will study their Ferrers diagrams.

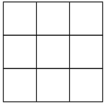
6+5+1



7+6+4+3

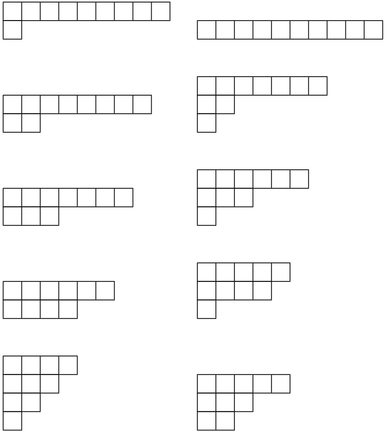


3+3+3



Distinct Parts and Ferrers Diagrams

Here are all the Ferrers diagrams for the partitions of 10 into distinct parts.



The left (resp. right) column contains all the partitions of 10 into distinct parts where the number of parts is even (resp. odd).

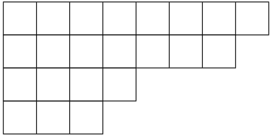
$$d_e(10) - d_o(10) = 0$$

Next, we will see that this difference is almost always zero and we will determine when it is not zero.

We will achieve this by systematically and uniquely pairing these partitions.

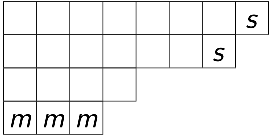
Distinct Parts and Ferrers Diagrams

Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)



Distinct Parts and Ferrers Diagrams

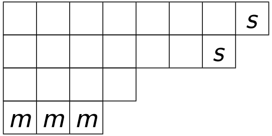
Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)



Let m to be the smallest part of the partition and let s be the right-most diagonal ($m = 3, s = 2$).

Distinct Parts and Ferrers Diagrams

Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)

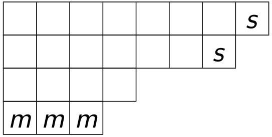


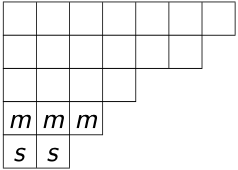
Let m to be the smallest part of the partition and let s be the right-most diagonal ($m = 3, s = 2$).

Rule: If $m > s$, move the right-most diagonal to make a new smallest part of the partition. Otherwise $m \leq s$, move the smallest part to the top of the right-most diagonal.

Distinct Parts and Ferrers Diagrams

Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)



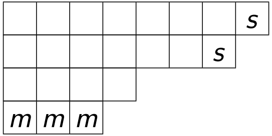


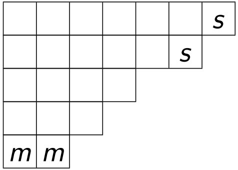
Let m to be the smallest part of the partition and let s be the right-most diagonal ($m = 3, s = 2$).

Rule: If $m > s$, move the right-most diagonal to make a new smallest part of the partition. Otherwise $m \leq s$, move the smallest part to the top of the right-most diagonal.

Distinct Parts and Ferrers Diagrams

Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)



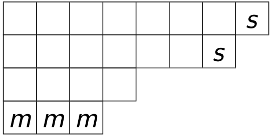


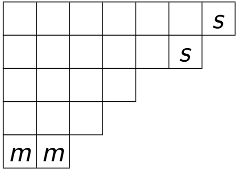
Let m to be the smallest part of the partition and let s be the right-most diagonal ($m = 3, s = 2$).

Rule: If $m > s$, move the right-most diagonal to make a new smallest part of the partition. Otherwise $m \leq s$, move the smallest part to the top of the right-most diagonal.

Distinct Parts and Ferrers Diagrams

Consider an arbitrary partition into distinct parts. (Ex. 8+7+4+3)





Let m to be the smallest part of the partition and let s be the right-most diagonal ($m = 3, s = 2$).

Rule: If $m > s$, move the right-most diagonal to make a new smallest part of the partition. Otherwise $m \leq s$, move the smallest part to the top of the right-most diagonal.

Does this map always pair partitions into distinct parts with opposing parities?

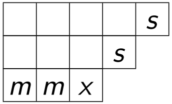
Example. This map fails for the partitions 5+4+3 and 6+5+4.

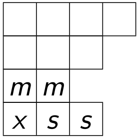
Distinct Parts and Ferrers Diagrams

Example. This map fails for the partitions 5+4+3 and 6+5+4.

				s
			s	
m	m	x		

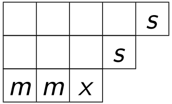
Example. This map fails for the partitions 5+4+3 and 6+5+4.

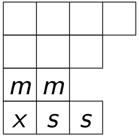


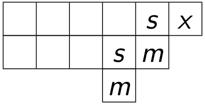


Distinct Parts and Ferrers Diagrams

Example. This map fails for the partitions 5+4+3 and 6+5+4.

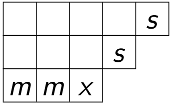


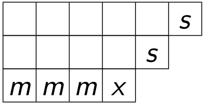


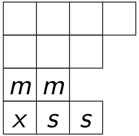


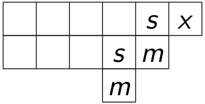
Distinct Parts and Ferrers Diagrams

Example. This map fails for the partitions 5+4+3 and 6+5+4.

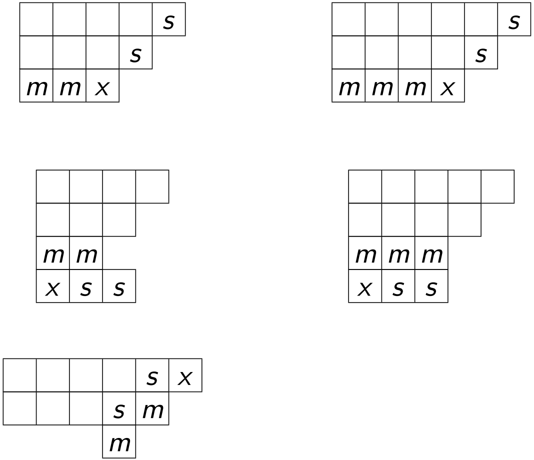




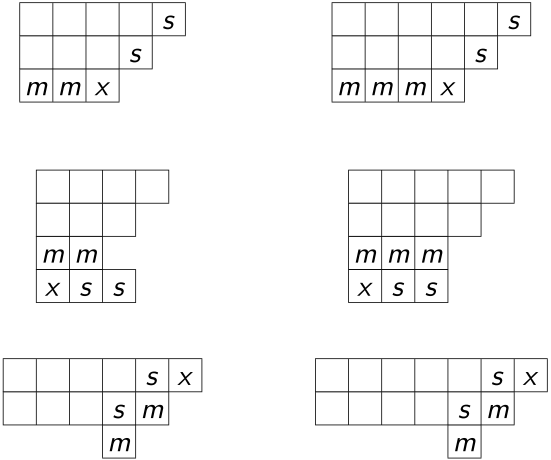




Example. This map fails for the partitions 5+4+3 and 6+5+4.

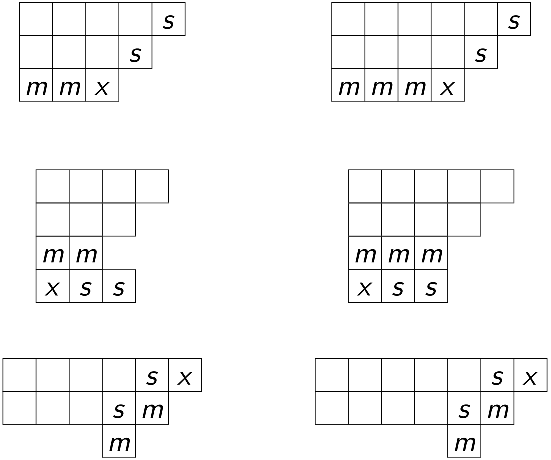


Example. This map fails for the partitions 5+4+3 and 6+5+4.



Distinct Parts and Ferrers Diagrams

Example. This map fails for the partitions 5+4+3 and 6+5+4.



We will have a problem if and only if:

- the bottom-right-most cell is counted by m and s, and
- $m = s$ or $m = s + 1$.

Hence, $d_e(n) - d_o(n) = 0$ unless n has a partition into distinct parts whose Ferrers diagram has this shape. If this happens, the parity balance will be off by 1.

Pentagonal Numbers

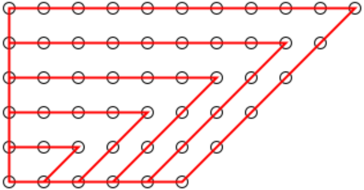
What exactly is this shape of Ferrers diagram?

Pentagonal Numbers

What exactly is this shape of Ferrers diagram? A pentagon!

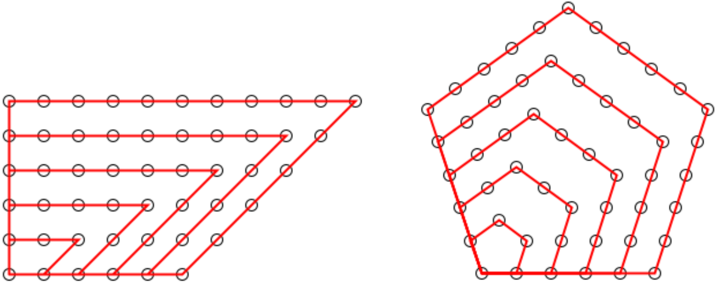
Pentagonal Numbers

What exactly is this shape of Ferrers diagram? A pentagon!



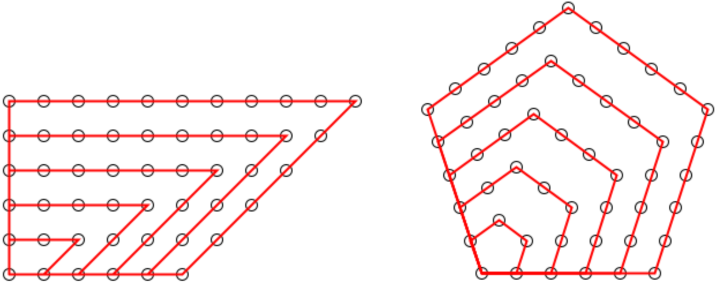
Pentagonal Numbers

What exactly is this shape of Ferrers diagram? A pentagon!



Pentagonal Numbers

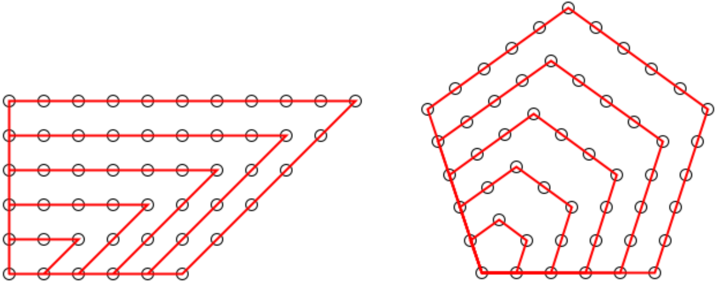
What exactly is this shape of Ferrers diagram? A pentagon!



Then k-th pentagonal numbers are $\frac{k(3k-1)}{2}$ and $\frac{k(3k+1)}{2}$ (the $m = s$ and $m = s + 1$ cases respectively).

Pentagonal Numbers

What exactly is this shape of Ferrers diagram? A pentagon!



Then k-th pentagonal numbers are $\frac{k(3k-1)}{2}$ and $\frac{k(3k+1)}{2}$ (the $m = s$ and $m = s + 1$ cases respectively). The corresponding partitions have k parts.

Pentagonal Numbers

The k-th pentagonal numbers are $\frac{k(3k-1)}{2}$ and $\frac{k(3k+1)}{2}$.

Pentagonal Numbers

The k-th pentagonal numbers are $\frac{k(3k-1)}{2}$ and $\frac{k(3k+1)}{2}$.

Review. We paired partitions, showing that the n-th coefficient in

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=1}^{\infty}(d_e(n) - d_o(n))x^n$$

is zero except for when n is a (generalized) pentagonal number.

Pentagonal Numbers

The k-th pentagonal numbers are $\frac{k(3k-1)}{2}$ and $\frac{k(3k+1)}{2}$.

Review. We paired partitions, showing that the n-th coefficient in

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=1}^{\infty}(d_e(n) - d_o(n))x^n$$

is zero except for when n is a (generalized) pentagonal number.

Theorem (Pentagonal Number Theorem)

For $|x| < 1$,

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=1}^{\infty}(-1)^k(x^{k(3k-1)/2} + x^{k(3k+1)/2})$$

$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + \cdots .$$

Pentagonal Numbers

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + +x^{22} + x^{26} + \cdots$$

Pentagonal Numbers

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + +x^{22} + x^{26} + \cdots$$

Define a_m to be the m-th coefficient in this power series.

$$a_m = d_e(m) - d_o(m)$$

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + +x^{22} + x^{26} + \cdots$$

Define a_m to be the m-th coefficient in this power series.

$$a_m = d_e(m) - d_o(m)$$

We defined the above series to be the reciprocal of the partition generating function. Hence

$$\left(\sum_{m=0}^{\infty} a_m x^m\right)\left(\sum_{n=0}^{\infty} p(n)x^n\right) = 1.$$

Pentagonal Numbers

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + +x^{22} + x^{26} + \cdots$$

Define a_m to be the m-th coefficient in this power series.

$$a_m = d_e(m) - d_o(m)$$

We defined the above series to be the reciprocal of the partition generating function. Hence

$$\left(\sum_{m=0}^{\infty} a_m x^m\right)\left(\sum_{n=0}^{\infty} p(n) x^n\right) = 1.$$

After some series manipulation, we can arrive at the desired recursive formula for $p(n)$.

Pentagonal Numbers

$$
\begin{aligned}
1 &= \left(\sum_{m=0}^{\infty} a_m x^m\right)\left(\sum_{n=0}^{\infty} p(n) x^n\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m p(n) x^{m+n} \\
&= \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} a_m p(k-m) x^k \\
&= \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} a_m p(k-m)
\end{aligned}
$$

Pentagonal Numbers

$$1 = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} a_m p(k-m)$$

Matching coefficients, we see $1 = p(0)$ and for $k \geq 1$,

$$0 = \sum_{m=0}^{k} a_m p(k-m).$$

Pulling the first term off the sum,

$$\begin{array}{lllllll} p(n) = & +p(n-1) & +p(n-2) & -p(n-5) & -p(n-7) & +p(n-12) & + \cdots \\ \prod_{n=1}^{\infty}(1-x^n) = 1 & -x & -x^2 & +x^5 & +x^7 & -x^{12} & + \cdots \end{array}$$

Pentagonal Numbers

$$1 = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} a_m p(k-m)$$

Matching coefficients, we see $1 = p(0)$ and for $k \geq 1$,

$$0 = \sum_{m=0}^{k} a_m p(k-m).$$

Pulling the first term off the sum,

$$\begin{aligned} p(n) &= \quad +p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots \\ \prod_{n=1}^{\infty}(1-x^n) &= 1 \quad -x \quad -x^2 \quad +x^5 \quad +x^7 \quad -x^{12}+\cdots \end{aligned}$$

Closing Remarks

- The study of integer partitions is an approachable subject.
- It's also a very deep/rich subject.

Closing Remarks

- The study of integer partitions is an approachable subject.
- It's also a very deep/rich subject.

Exercises. Define the *conjugation map* on partitions by reflecting their Ferrers diagram across the diagonal.

- $p(n \mid \text{largest part } m) = p(n \mid \text{m parts})$.
- $p(n \mid \text{self-conjugate}) = p(n \mid \text{distinct odd parts})$

Show this one in two ways. Describe a bijection between the sets, and show their generating functions are the same.

- $p(n \mid \text{distinct parts}) = p(n \mid \text{odd parts})$.

Closing Remarks

- The study of integer partitions is an approachable subject.
- It's also a very deep/rich subject.

Exercises. Define the *conjugation map* on partitions by reflecting their Ferrers diagram across the diagonal.

- $p(n \mid \text{largest part } m) = p(n \mid \text{m parts})$.
- $p(n \mid \text{self-conjugate}) = p(n \mid \text{distinct odd parts})$

Show this one in two ways. Describe a bijection between the sets, and show their generating functions are the same.

- $p(n \mid \text{distinct parts}) = p(n \mid \text{odd parts})$.

Thank you.